

YNU-HEPTh-01-102  
CPT-2001/P.4242  
KUCP-195  
October 2001

# Positivity Constraints on Photon Structure Functions

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## Abstract

We investigate the positivity constraints for the structure functions of both virtual and real photon. From the Cauchy-Schwarz inequality we derive three positivity conditions for the general virtual photon case, which reduce, in the real photon case, to one condition relating the polarized and unpolarized structure functions.

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The photon structure has been studied through the two-photon processes in  $e^+e^-$  collisions as well as the resolved photon processes in the electron-proton collider. Based on the perturbative QCD (pQCD), the unpolarized parton distributions in the photon have been extracted from the measured structure function  $F_2^\gamma$  [1]. Recently there has been growing interest in the study of polarized photon structure functions [2, 3]. Especially the first moment of the spin-dependent structure function  $g_1^\gamma$  has attracted much attention in the literature in connection with its relevance for the axial anomaly [4, 5, 6, 7, 8]. The next-to-leading order QCD analysis of  $g_1^\gamma$  has been performed in the literature [9, 10, 11]. There exists a positivity bound,  $|g_1^\gamma| \leq F_1^\gamma$ , which comes out from the definition of structure functions,  $g_1^\gamma$  and  $F_1^\gamma$ , and positive definiteness of the  $s$ -channel helicity-nonflip amplitudes. This bound was closely analyzed recently [11].

In the case of virtual photon target, there appear eight structure functions [12, 13, 14], most of which have not been measured yet and, therefore, unknown. In a situation like this, positivity would play an important role in constraining these unknown structure functions. It is well known in the deep inelastic scattering off nucleon that various bounds have been obtained for the spin-dependent observables and parton distributions in a nucleon by means of positivity conditions [15].

In the present paper we investigate the model-independent constraints for the structure functions of virtual (off-shell) and real (on-shell) photon target. We obtain three positivity conditions for the virtual photon case and one condition for the real photon, the latter of which relates the polarized and unpolarized structure functions.

Let us consider the virtual photon-photon forward scattering:  $\gamma(q) + \gamma(p) \rightarrow \gamma(q) + \gamma(p)$  illustrated in Fig.1. The  $s$ -channel helicity amplitudes are given by

$$W(ab|a'b') = \epsilon_\mu^*(a)\epsilon_\rho^*(b)W^{\mu\nu\rho\tau}\epsilon_\nu(a')\epsilon_\tau(b') , \quad (1)$$

where  $p$  and  $q$  are four-momenta of the target and probe photon, respectively,  $\epsilon_\mu(a)$  represents the photon polarization vector with helicity  $a$ , and  $a, a' = 0, \pm 1$ , and  $b, b' = 0, \pm 1$ . Due to the angular momentum conservation,  $W(ab|a'b')$  vanishes unless it satisfies the condition  $a - b = a' - b'$ . And parity conservation and time reversal invariance lead to the following properties for  $W(ab|a'b')$  [16]:

$$W(ab|a'b') = W(-a, -b | -a', -b') \quad \text{Parity conservation ,}$$

$$= W(a'b'|ab) \quad \text{Time reversal invariance} . \quad (2)$$

Thus in total we have eight independent  $s$ -channel helicity amplitudes, which we may take as  $W(1,1|1,1)$ ,  $W(1,-1|1,-1)$ ,  $W(1,0|1,0)$ ,  $W(0,1|0,1)$ ,  $W(0,0|0,0)$ ,  $W(1,1|-1,-1)$ ,  $W(1,1|0,0)$ , and  $W(1,0|0,-1)$ . The first five amplitudes are helicity-nonflip and the rest are helicity-flip. It is noted that  $s$ -channel helicity-nonflip amplitudes are semi-positive, but not the helicity-flip ones. And corresponding to these three helicity-flip amplitudes, we will obtain three non-trivial positivity constraints.

The helicity amplitudes may be expressed in terms of the transition matrix elements from the state  $|a, b\rangle$  of two virtual photons with helicities  $a$  and  $b$ , to the unobserved state  $|X\rangle$  as

$$\begin{aligned} W(ab|ab) &= \sum_X |\langle X|a, b\rangle|^2, \\ W(ab|a'b') &= \text{Re} \sum_X \langle X|a, b\rangle^* \langle X|a', b'\rangle \quad (a \neq a', b \neq b') . \end{aligned} \quad (3)$$

Then, a Cauchy-Schwarz inequality [17, 18]

$$\sum_X |\langle X|a, b\rangle + \alpha \langle X|a', b'\rangle|^2 \geq 0 , \quad (4)$$

which holds for an arbitrary real number  $\alpha$ , leads to a positivity bound for the helicity amplitudes:  $|W(a, b|a', b')| \leq \sqrt{W(a, b|a, b)W(a', b'|a', b')}$ . Writing down explicitly, we obtain the following three positivity constraints:

$$|W(1, 1|-1, -1)| \leq W(1, 1|1, 1) , \quad (5)$$

$$|W(1, 1|0, 0)| \leq \sqrt{W(1, 1|1, 1)W(0, 0|0, 0)} , \quad (6)$$

$$|W(1, 0|0, -1)| \leq \sqrt{W(1, 0|1, 0)W(0, 1|0, 1)} . \quad (7)$$

In terms of the eight independent amplitudes introduced by Budnev, Chernyak and Ginzburg [12], the above three conditions can be rewritten as

$$|W_{\text{TT}}^\tau| \leq (W_{\text{TT}} + W_{\text{TT}}^a) , \quad (8)$$

$$|W_{\text{TS}}^\tau + W_{\text{TS}}^{\tau a}| \leq \sqrt{(W_{\text{TT}} + W_{\text{TT}}^a)W_{\text{SS}}} , \quad (9)$$

$$|W_{\text{TS}}^\tau - W_{\text{TS}}^{\tau a}| \leq \sqrt{W_{\text{TS}}W_{\text{ST}}} , \quad (10)$$

where T and S refer to the transverse and longitudinal photon, respectively, and the superscripts "τ" and "a" imply the relevance to the helicity-flip amplitudes and polarized ones, respectively.

For the real photon,  $p^2 = 0$ , the number of independent helicity amplitudes reduces to four. They are  $W(1, 1|1, 1)$ ,  $W(1, -1|1, -1)$ ,  $W(0, 1|0, 1)$ , and  $W(1, 1| -1, -1)$ , which are related to four structure functions  $W_i^\gamma$  as follows [12, 13, 14, 19]:

$$\begin{aligned}\frac{1}{2} [W(1, 1|1, 1) + W(1, -1|1, -1)] &= W_1^\gamma , \\ W(0, 1|0, 1) &= -W_1^\gamma + \frac{(p \cdot q)^2}{Q^2} W_2^\gamma , \\ \frac{1}{2} W(1, 1| -1, -1) &= W_3^\gamma , \\ \frac{1}{2} [W(1, 1|1, 1) - W(1, -1|1, -1)] &= W_4^\gamma ,\end{aligned}\tag{11}$$

where the last one is the polarized structure function and usually denoted by  $g_1^\gamma$  with  $W_4^\gamma = \frac{1}{2}g_1^\gamma$ . Also the first one,  $W_1^\gamma$ , is often referred to as  $F_1^\gamma$  with  $W_1^\gamma = \frac{1}{2}F_1^\gamma$ .

For the real photon case we have only one constraint, i.e., the first inequality (5), which is rewritten as

$$2|W_3^\gamma| \leq (W_1^\gamma + W_4^\gamma) .\tag{12}$$

It is interesting to recall that the polarized structure function  $W_4^\gamma$  of the real photon satisfies a remarkable sum rule [4, 5, 6, 7, 8]

$$\int_0^1 W_4^\gamma(x, Q^2) dx = 0 .\tag{13}$$

The integral of  $|W_3^\gamma|$  is, therefore, bounded from above by the first moment of  $W_1^\gamma$ ,

$$\int_0^1 |W_3^\gamma(x, Q^2)| dx \leq \frac{1}{2} \int_0^1 W_1^\gamma(x, Q^2) dx .\tag{14}$$

Now let us examine whether the inequality (12) is actually satisfied or not by the structure functions obtained in the simple parton model (PM). By evaluating the box (a massive quark-loop) diagrams with  $p^2 = 0$ , ignoring the power correction of  $m^2/Q^2$  with quark mass  $m$ , the photon structure functions have been obtained as follows:

$$W_1^\gamma(x, Q^2)_{\text{PM}} = \frac{\alpha}{2\pi} \delta_\gamma \left\{ [x^2 + (1-x)^2] \ln \left( \frac{Q^2}{m^2} \frac{1-x}{x} \right) - 1 + 4x(1-x) \right\} ,$$

$$\begin{aligned}
W_3^\gamma(x, Q^2)_{\text{PM}} &= \frac{\alpha}{2\pi} \delta_\gamma(-x^2) , \\
W_4^\gamma(x, Q^2)_{\text{PM}} &= \frac{\alpha}{2\pi} \delta_\gamma \left\{ (2x-1) \ln \left( \frac{Q^2}{m^2} \frac{1-x}{x} \right) + 3 - 4x \right\} ,
\end{aligned} \tag{15}$$

where  $x = Q^2/(2p \cdot q)$ ,  $\alpha = e^2/4\pi$ , the QED coupling constant, and  $\delta_\gamma = 3 \sum_{i=1}^{N_f} e_i^4$ , with  $N_f$ , the number of the active flavors. Using these expressions, we examine the constraint (12) numerically and find that it is satisfied almost all allowed region of  $x$  except near the limit  $x \rightarrow x_{\text{max}} = 1/(1 + 4m^2/Q^2)$ . However, the violation of the inequality near  $x_{\text{max}}$  is an artifact, since the limiting procedures of  $Q^2 \rightarrow \infty$  and  $x \rightarrow x_{\text{max}}$  are not exchangeable. In fact, the exact PM calculation of  $W_i^\gamma$ 's with  $Q^2$  kept finite gives

$$\begin{aligned}
W_1^\gamma|_{\text{PM}} &= \frac{\alpha}{2\pi} \delta_\gamma \left\{ \left( \ln \frac{1+\beta}{1-\beta} \right) \left[ x^2 + (1-x)^2 - 8x^2 \frac{m^4}{Q^4} - 4(x^2-x) \frac{m^2}{Q^2} \right] \right. \\
&\quad \left. + \beta \left[ 4x(1-x) - 1 + 4(x^2-x) \frac{m^2}{Q^2} \right] \right\} , \\
W_3^\gamma|_{\text{PM}} &= -\frac{\alpha}{2\pi} \delta_\gamma \left\{ \left( \ln \frac{1+\beta}{1-\beta} \right) \left[ 4x^2 \frac{m^4}{Q^4} + 4x^2 \frac{m^2}{Q^2} \right] + \beta \left[ x^2 + 2(x-x^2) \frac{m^2}{Q^2} \right] \right\} , \\
W_4^\gamma|_{\text{PM}} &= \frac{\alpha}{2\pi} \delta_\gamma \left\{ \left( \ln \frac{1+\beta}{1-\beta} \right) (2x-1) + \beta [-4x+3] \right\} ,
\end{aligned} \tag{16}$$

where  $\beta = \sqrt{1 - \frac{4m^2x}{Q^2(1-x)}}$ . The expression of  $W_4^\gamma|_{\text{PM}}$  is consistent with the result of Ref.[20], where polarized gluon structure functions were considered. The results for  $W_1^\gamma|_{\text{PM}}$  and  $W_3^\gamma|_{\text{PM}}$  in (16) are new. It is noted that since  $\beta \rightarrow 0$  for  $x \rightarrow x_{\text{max}}$ , all  $W_1^\gamma|_{\text{PM}}$ ,  $W_3^\gamma|_{\text{PM}}$ , and  $W_4^\gamma|_{\text{PM}}$  vanish at  $x = x_{\text{max}}$ . Using these exact PM results in (16), we find numerically that the inequality (12) is indeed satisfied for all allowed region of  $x$ . Moreover, once expressed as functions of  $x$  and  $\beta$ , the helicity-nonflip amplitudes  $W(1, 1|1, 1)|_{\text{PM}}$  and  $W(1, -1|1, -1)|_{\text{PM}}$  are easily shown to be non-negative for  $0 \leq \beta < 1$ , and  $0 \leq x < 1$ , as they should be. On the other hand, the helicity-flip amplitude  $W(1, 1|-1, -1)|_{\text{PM}}$  turns out to be negative.

As stated earlier, in the case of virtual photon,  $p^2 = -P^2 \neq 0$ , there appear eight structure functions (four of them are new) and we have derived three positivity constraints on these functions. But up to now little attention has been paid to the virtual photon case and, therefore, we have slight knowledge of the new photon

structure functions. In this situation it is worthwhile to investigate these new structure functions in the simple PM and examine that the three positivity constraints (8)-(10) actually hold [22].

Especially, in the kinematical region,  $\Lambda^2 \ll P^2 \ll Q^2$ , where the mass squared of the target photon ( $P^2$ ) is much bigger than the QCD scale parameter ( $\Lambda^2$ ), some of the photon structure functions are predictable in pQCD entirely up to the next-leading-order (NLO), since the hadronic component on the photon can also be dealt with perturbatively. Following this strategy, the virtual photon structure functions, unpolarized  $F_2^\gamma(x, Q^2, P^2)$  and  $F_L^\gamma(x, Q^2, P^2)$  [21] and polarized  $g_1^\gamma(x, Q^2, P^2)$  [10], were studied up to the NLO. Since  $F_1^\gamma \equiv (F_2^\gamma - F_L^\gamma)/x = 2W_1^\gamma$  and  $g_1^\gamma = 2W_4^\gamma$ , it is also interesting to see if the inequality (12) is satisfied by the pQCD results for the above kinematical region [22]. The virtual photon structure function  $W_3^\gamma(x, Q^2, P^2)$  is expected to be given by the same expression as the PM result (15) up to  $\mathcal{O}(1/\ln(Q^2/\Lambda^2))$ , since there exist no twist-2 quark operators contributing to  $W_3^\gamma$  [19, 23].

So far we have only considered the constraints on the structure functions. Now our argument can be extended to the quark contents of the photon, for which we can also write down inequalities involving various distributions. Following Ref. [17], let us define the helicity amplitudes given by

$$\mathcal{W}(ab|ab) \equiv \sum_X \langle \gamma_b | O^\dagger | q_a, X \rangle \langle X, q_a | O | \gamma_b \rangle, \quad (17)$$

$$\mathcal{W}(ab|a'b') \equiv \text{Re} \sum_X \langle \gamma_b | O^\dagger | q_a, X \rangle \langle X, q_{a'} | O | \gamma_{b'} \rangle \quad (a \neq a', b \neq b'), \quad (18)$$

where all the suffices,  $a, b, a'$  and  $b'$ , refer to the helicities of the quarks and virtual photons, and  $O$ 's denote bilinear quark operators. One also has to sum over all intermediate states  $X$ . Then we can derive, in a similar fashion based on the Cauchy-Schwarz inequality,

$$|\mathcal{W}(ab|a'b')| \leq \sqrt{\mathcal{W}(ab|ab)\mathcal{W}(a'b'|a'b')}. \quad (19)$$

In our present case, the above helicity amplitudes become nothing but the following quark distributions:

$$q_\gamma^\pm = \sum_X \langle \gamma_+ | O^\dagger | q_\pm, X \rangle \langle X, q_\pm | O | \gamma_+ \rangle, \quad (20)$$

$$q_\gamma^0 = \sum_X \langle \gamma_0 | O^\dagger | q_+, X \rangle \langle X, q_+ | O | \gamma_0 \rangle , \quad (21)$$

$$h_\gamma^q = \text{Re} \sum_X \langle \gamma_+ | O^\dagger | q_+, X \rangle \langle X, q_- | O | \gamma_0 \rangle , \quad (22)$$

where  $q_\gamma^\pm$  ( $q_\gamma^0$ ) denotes the longitudinally (transversely) polarized quark distribution inside the photon, and  $h_\gamma^q$  is the (chirality-odd) transversity distribution, the photon analog of  $h_1^q$  for the nucleon case. Here we note that the photon structure function  $F_1^\gamma$  ( $g_1^\gamma$ ) can be expressed as a sum over the active quark (or anti-quark) distributions  $q_\gamma$  ( $\Delta q_\gamma$ ), with  $q_\gamma = q_\gamma^+ + q_\gamma^-$  and  $\Delta q_\gamma = q_\gamma^+ - q_\gamma^-$ . Now taking  $a = 1/2$ ,  $b = 1$ ,  $a' = -1/2$ ,  $b' = 0$  in (19) we get

$$|h_\gamma^q| \leq \sqrt{q_\gamma^+ q_\gamma^0} . \quad (23)$$

Hence we have the following positivity condition for the transversity distribution  $h_\gamma^q$ ,

$$|h_\gamma^q| \leq \sqrt{\left(\frac{q_\gamma + \Delta q_\gamma}{2}\right) \cdot q_\gamma^0} . \quad (24)$$

This is an extension of the inequality obtained for the nucleon case [15, 17, 18]. The transversity distribution  $h_\gamma^q$  of the photon could be measured by the semi-inclusive process in the two-photon reactions provided by the future polarized  $e^+e^-$  collision experiments.

In summary we have investigated the model-independent positivity constraints for the photon structure functions which could be studied in future experiments. We also discussed a positivity bound for the quark distributions relevant for the spin-dependent semi-inclusive process in two-photon reactions. We expect these bounds would provide useful constraints for studying the yet unknown polarized and unpolarized photon structures.

## Acknowledgement

We thank Werner Vogelsang and Hideshi Baba for useful discussions. This work is partially supported by the Grant-in-Aid from the Japan Ministry of Education and Science, No.(C)(2)-12640266.



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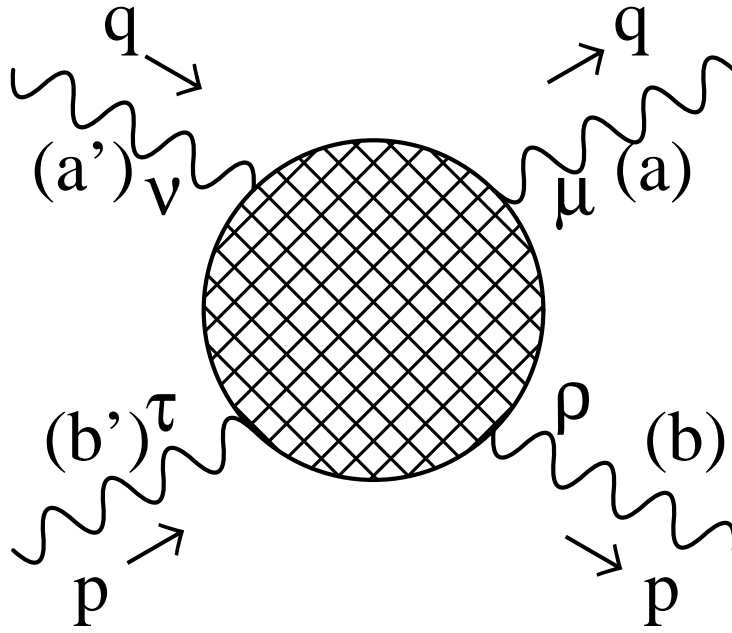


Figure Captions

**Fig.1**

Virtual photon-photon forward scattering with momenta  $q(p)$  and helicities  $a(b)$  and  $a'(b')$ .

